Gram-Schmidt Method

↪ Finds orthogonal bases from a group of basis vectors

$$U_k = V_k - \sum_{j=1}^{k-1} \mathrm{Proj}_{u_j}(V_k)$$

Orthogonal base

o.g. vector

Gets rid of any // components to all other established bases (U_k)

⇒ We are given $(V_1, V_2, \ldots V_k)$ and turning them into the orthogonal $(U_1, U_2, \ldots, U_j)$

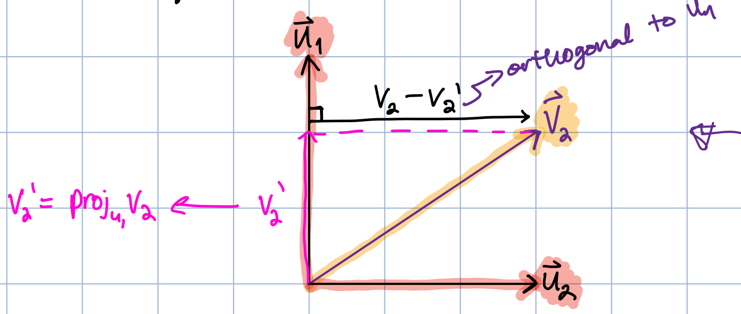
Step 1:

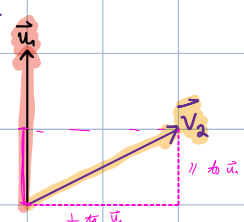
⇒ Take V_1 & make it the first "established base"

i.e. $U_1 = V_1$

Step 2:

⇒ Find the component of V_2 that is orthogonal to U_1 to find U_2 (converting $V_2 \to U_2$)



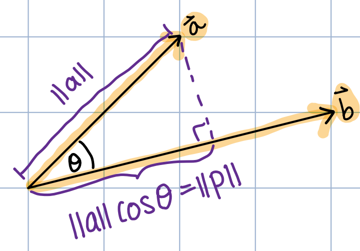


i.e. ⇒ $U_2 = V_2 - V_2'$

$U_2 = V_2 - \mathrm{proj}_{u_1} V_2$

(Basic shit I know)

projections & dot products:



Dot Product:

Scalar

is the projection of one vector onto another - Scalar

⇒ Base × height

⇒ $\|B\| \cdot \|a\| \cos\theta$

Dot product Formula

⇒ $a \cdot b = \|a\| \cdot \|b\| \cos\theta$

⇒ $\cos\theta = \frac{adj}{hyp} = \frac{\|P\|}{\|a\|}$

⇒ $\cos\theta = \cos\theta$

$$\frac{a \cdot b}{\cancel{\|a\|} \cdot \|b\|} = \frac{\|P\|}{\cancel{\|a\|}}$$

Projection of a onto b → ⇒ $\|P\| = \frac{a \cdot b}{\|b\|}$

Now, getting the vector:

→ In direction $\hat{b}$

→ $\hat{b} = \frac{\vec{b}}{\|b\|}$

⇒ $\|P\| \cdot \hat{b} = \frac{a \cdot b}{\|b\|} \cdot \frac{\vec{b}}{\|b\|}$

Vector projection

⇒ $\vec{P} = \frac{a \cdot b}{\|b\|^2} \cdot \vec{b}$

Step 3:

⇒ Continue, taking away any // components to already established orthogonal bases

e.g.: $U_3 = V_3 - \mathrm{proj}_{u_2} V_3 - \mathrm{proj}_{u_1} V_3$

Example:

EXAMPLE 7 Applying the Gram-Schmidt Orthonormalization Process

Apply the Gram-Schmidt orthonormalization process to the following basis for R^3.

$$B = \{\overset{\mathbf{v}_1}{(1, 1, 0)}, \overset{\mathbf{v}_2}{(1, 2, 0)}, \overset{\mathbf{v}_3}{(0, 1, 2)}\}$$

SOLUTION

Applying the Gram-Schmidt orthonormalization process produces

$$\mathbf{w}_1 = \mathbf{v}_1 = (1, 1, 0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (1, 2, 0) - \frac{3}{2}(1, 1, 0) = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2$$

$$= (0, 1, 2) - \frac{1}{2}(1, 1, 0) - \frac{1/2}{1/2}\left(-\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$= (0, 0, 2).$$

The set $B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis for R^3. Normalizing each vector in B' produces

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1).$$

So, $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3.

Given $V_1 = (1,1,0)^T$; $V_2 = (1,2,0)^T$; $V_3 = (0,1,2)^T$

$\Rightarrow u_1 = V_1$ $\qquad proj_b\, a = \frac{a \cdot b}{\|b\|^2} \cdot \vec{b}$

$$\Rightarrow \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow u_2 = V_2 - Proj_{u_1} V_2$$

$$= V_2 - \frac{V_2 \cdot u_1}{\|u_1\|^2} \cdot \vec{u}_1 \implies V_2 \cdot u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 + 2 + 0 = 3$$

$$\|u_1\|^2 = 1^2 + 1^2 = 2$$

$$\Rightarrow \vec{u}_2 = \vec{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}} - \left(\frac{3}{2}\right) \cdot \vec{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$$

$$\vec{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \vec{u}_2 = \begin{pmatrix} \frac{-1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow u_3 = \vec{V}_3 - Proj_{u_1} V_3 - Proj_{u_2} V_3$$

$$= \vec{V}_3 - \frac{\overset{①}{V_3 \cdot u_1}}{\|u_1\|^2} \cdot \vec{u}_1 - \frac{\overset{②}{V_3 \cdot u_2}}{\|u_2\|^2} \cdot \vec{u}_2$$

$$① \; V_3 \cdot u_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 + 1 + 0 = 1$$

$$\|u_1\|^2 = 2$$

$$② \; V_3 \cdot u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = \frac{1}{2}$$

$$\|u_2\| = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \vec{u}_3 = \vec{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}} - \left(\frac{1}{2}\right) \vec{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} - \left(\cancel{\frac{\frac{1}{2}}{\frac{1}{2}}}\right)^{1} \vec{\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}}$$

$$\Rightarrow \vec{u}_3 = \begin{pmatrix} 0 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} \\ \cancel{1} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} \\ 2 - 0 - 0 \end{pmatrix}$$

$$\Rightarrow \vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Now, Make them **Orthonormal**:

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

$$\Rightarrow \hat{u}_1 = \frac{\vec{u}_1}{\|\vec{u}_1\|} \quad \rightarrow \|\vec{u}_1\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow \hat{u}_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \hat{u}_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \cdot \frac{1}{\frac{1}{\sqrt{2}}} \quad \leftarrow \|\vec{u}_2\| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \hat{u}_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$$

$$\Rightarrow \hat{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \frac{1}{2} \quad \leftarrow \|\vec{u}_3\| = \sqrt{4} = 2$$

$$\Rightarrow \hat{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$